

On the global regularity for nonlinear systems of the p-Laplacian type

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Abstract

We consider the Dirichlet boundary value problem for nonlinear systems of partial differential equations with p -structure. We choose two representative cases: the “full gradient case”, corresponding to a p-Laplacian, and the “symmetric gradient case”, arising from mathematical physics. The domain is either the so called “cubic domain” or a bounded open subset of \mathbb{R}^3 with a smooth boundary. We are interested in regularity results, up to the boundary, for the second order derivatives of the velocity field. Depending on the model and on the range of p , $p < 2$ or $p > 2$, we prove different regularity results. It is worth noting that in the full gradient case, with $p < 2$, we cover the degenerate case, and obtain $W^{2,q}$ -global regularity results, for arbitrarily large values of q .

Keywords: p-Laplacian systems, regularity up to the boundary, full regularity.

1 Introduction

We are concerned with the regularity problem for solutions of nonlinear systems of partial differential equations with p -structure, $p > 1$, under Dirichlet boundary conditions. In order to emphasize the main ideas we confine ourselves to the following representative cases (where $\mu \geq 0$ is a fixed constant):

The “full gradient case”

$$(1.1) \quad -\nabla \cdot S(\nabla u) = f,$$

where

$$(1.2) \quad S(\nabla u) = (\mu + |\nabla u|)^{p-2} \nabla u;$$

and the “symmetric gradient case”

$$(1.3) \quad -\nabla \cdot S(\mathcal{D}u) = f,$$

where

$$(1.4) \quad S(\mathcal{D}u) = (\mu + |\mathcal{D}u|)^{p-2} \mathcal{D}u.$$

As usual,

$$\mathcal{D}u = \frac{1}{2}(\nabla u + \nabla u^T)$$

is the symmetric part of the gradient of u .

When $\mu = 0$ in (1.2), the system (1.1) is the well-known p-Laplacian system.

It is worth noting that our results concern global (up to the boundary), full regularity for the second derivatives of solutions to the previous systems, with Dirichlet boundary conditions (one could also consider slip type boundary conditions). The regularity issue for systems like (1.1) has received substantial attention, mostly concerned with an equation in place of a system, and with $C_{loc}^{1,\alpha}$ -regularity. In the scalar case, existence and interior integrability of the second derivatives are shown in [32], for any $p > 1$; in [27] the regularity up to the boundary is obtained for any $p \in (1, 2)$. For systems (solutions are N -dimensional vector fields, $N > 1$), we recall [1] for $p \in (1, 2)$, [20] and [33] for $p > 2$, and [22] for any $p > 1$. These papers deal only with homogeneous systems and the techniques, sometimes quite involved, seem not to be directly applicable to the non-homogeneous setting. In particular, [1] is the only paper in which the L^2 -regularity of second derivatives is considered. However, the results are shown only in the interior. Therefore our results seem to be the first regularity results, up to the boundary, for the second derivatives of solutions. Another main difference with the above papers is that we do not require differentiability of S , but merely Lipschitz continuity. For related results and for an extensive bibliography we also refer to papers [2], [14], [15], [17], [18], [25], [29], [30] and references therein.

We have not found papers dealing with the equations arising from the choice (1.4) for S . This kind of model is used in various branches of mathematical physics as, for instance, in non-linear elasticity or in non-linear diffusion. Actually, our interest in systems (1.1) and (1.3) arises from our previous studies on fluid dynamics problems. Indeed, we recall that a good model for non-Newtonian fluids with shear dependent viscosity is the following one

$$(1.5) \quad -\nabla \cdot [(\mu + |\mathcal{D}u|)^{p-2} \mathcal{D}u] + (u \cdot \nabla)u + \nabla \pi = f, \quad \nabla \cdot u = 0,$$

which can be obtained from (1.3) by adding the contribution of the pressure field π , the convective term $(u \cdot \nabla)u$ and the divergence free constraint. For this system, regularity up to the boundary has been considered in both the cases $p < 2$ and $p > 2$. The case $p = 2$ corresponds to the well known Navier-Stokes system for Newtonian fluids. For the more general regularity results and a wide bibliography on this topic, we refer the reader to [7], [9] for $p > 2$, and to [8] for $p < 2$. Despite many contributions to the regularity issue, $W^{2,2}$ -regularity up to the boundary for solutions to (1.5) is still open, even for the simplified setting of “generalized” Stokes system, obtained by dropping the convective term in (1.5). We mention the papers [12] and [13], which, as far as we know, are the only papers where the $W^{2,2}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ -regularity is obtained, under the additional assumption of a small force. The regularity proved below suggests that the main obstacle to the $W^{2,2}$ -regularity of solutions of (1.5) is actually the presence of the pressure term.

Our interest in fluid-mechanics, and in particular in non-Newtonian fluids, leads us to consider the case $n = N = 3$. However, it is worth noting that our results can be immediately extended to dimensions $n > 3$, and to N -dimensional vector fields, $N \neq 3$. Further, the explicit choices (1.2) and (1.4) are done in order to emphasize the core aspects of the results and to avoid additional technicalities. Therefore, we do not consider a more general dependence of S on ∇u or $\mathcal{D}u$, as for instance $S(\nabla u) = \varphi(|\nabla u|) \nabla u$, under suitable assumptions on the scalar function φ . For the same reason we avoid the introduction of lower order terms.

In the sequel we cover both the cases $p < 2$ and $p > 2$, with, however, some differences, and some restrictions on the exponent p , as follows:

Case $p < 2$: For $p < 2$ we consider the “full gradient case” (1.1). In this case, all results hold also in the degenerate case $\mu = 0$. For any bounded and sufficiently smooth domain Ω , we prove $W^{2,q}(\Omega)$ regularity, for any $q \geq 2$. Therefore, we get, as a by product, the Hölder continuity, up to the boundary, of the gradient of the solution. Results are obtained for p belonging to suitable intervals $[C, 2)$, where the constants C are defined precisely.

Case $p > 2$: We prove the $W^{2,2}$ -regularity in both cases, (1.1) and (1.3), provided that $\mu > 0$. We restrict our proofs to the “cubic domain case” (see the next section), where the interesting boundary condition (Dirichlet) is imposed on two opposite sides, and periodicity in the other two directions. This choice, introduced in reference [5] and used in a series of other papers (see for instance [4, 6, 10, 11]), is convenient in order to work with a flat boundary and, at the same time, with a bounded domain. The main reason is that, in proving the regularity theorem for $p > 2$ (see Theorem 2.1), we apply the difference quotients method: we appeal to translations parallel to the flat boundary, and then retrieve the normal derivatives from the equations. Then, the simplified framework of a cubic domain avoids the need of localization techniques and changes of variables. The results can be extended to smooth domains, by following [7], [8], and [9], where the extension is done for the more involved system of non-Newtonian fluids (see also [28]). See also the Remark 5.1.

2 Notation and statement of the main results

Throughout this paper we denote by Ω a bounded three-dimensional domain with smooth boundary, which we assume of class C^2 , and we consider the usual homogeneous Dirichlet boundary conditions

$$(2.1) \quad u|_{\partial\Omega} = 0.$$

Further, we denote by Q the cube $Q =]0, 1[^3$, and by Γ the two opposite faces of Q in the x_3 -direction, i.e.

$$\Gamma = \{x : |x_1| < 1, |x_2| < 1, x_3 = 0\} \cup \{x : |x_1| < 1, |x_2| < 1, x_3 = 1\}.$$

We impose the Dirichlet boundary conditions on Γ

$$(2.2) \quad u|_{\Gamma} = 0,$$

and periodicity, with period equal to 1, in both the x_1, x_2 directions.

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, m nonnegative integer and $p \in (1, +\infty)$, we denote the usual Lebesgue and Sobolev spaces, with the standard norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$, respectively. We usually denote the above norms by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, when the domain is clear. Further, we set $\|\cdot\| = \|\cdot\|_2$. We denote by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ and by $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, the strong dual of $W_0^{1,p}(\Omega)$ with norm $\|\cdot\|_{-1,p'}$. In notation concerning duality pairings, norms and functional spaces, we do not distinguish between scalar and vector fields.

We set

$$V_p(\Omega) = \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = 0\},$$

and

$$V_p(Q) = \{ v \in W^{1,p}(Q) : v|_{\Gamma} = 0, v \text{ is } x' - \text{periodic} \}.$$

By $V'_p(\Omega)$ and $V'_p(Q)$ we denote the dual spaces of $V_p(\Omega)$ and $V_p(Q)$, respectively.

We use the summation convention on repeated indexes, except for the index s . For any given pair of second order tensors B and C , we write $B \cdot C \equiv B_{ij} C_{ij}$.

We denote by the symbols c, c_1, c_2 , etc., positive constants that may depend on μ ; by capital letters, C, C_1, C_2 , etc., we denote positive constants independent of $\mu \geq 0$ (eventually, μ bounded from above). The same symbol c or C may denote different constants, even in the same equation.

We set $\partial_i u = \frac{\partial u}{\partial x_i}$, $\partial_{ij}^2 u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Moreover we set $(\nabla u)_{ij} = \partial_j u_i$ and $(\mathcal{D}u)_{ij} = \frac{1}{2}((\nabla u)_{ij} + (\nabla u)_{ji})$. We denote by $D^2 u$ the set of all the second partial derivatives of u . The symbol $D_*^2 u$ may denote any second-order partial derivative $\partial_{hk}^2 u$ except for the derivatives $\partial_{33}^2 u$. Moreover we set

$$(2.3) \quad |D^2 u|^2 := \sum_{i,j,k=1}^3 |\partial_{jk}^2 u_i|^2 \quad \text{and} \quad |D_*^2 u|^2 := \sum_{\substack{i,j,k=1 \\ (j,k) \neq (3,3)}}^3 |\partial_{jk}^2 u_i|^2.$$

We define the tensor $S(A)$ as

$$(2.4) \quad S(A) = (\mu + |A|)^{p-2} A,$$

with $\mu \geq 0$ fixed constant, $p > 1$, and A an arbitrary tensor field. It is easily seen that $S(A)$ satisfies the following property: there exists a positive constant C_1 such that

$$(2.5) \quad \frac{\partial S_{ij}(A)}{\partial A_{kl}} B_{ij} B_{kl} \geq C_1 (\mu + |A|)^{p-2} |B|^2,$$

for any tensor B . Further

$$(2.6) \quad (S(A) - S(B)) \cdot (A - B) \geq C_2 \frac{|A - B|^2}{(\mu + |A| + |B|)^{2-p}},$$

and

$$(2.7) \quad |S(A) - S(B)| \leq C_3 \frac{|A - B|}{(\mu + |A| + |B|)^{2-p}},$$

for any pair of tensors A and B , with C_2 and C_3 positive constants. The proof of the above estimates is essentially contained in [21]. We also refer to [16] for a detailed proof.

Our aim is to prove the regularity results up to the boundary given in the theorems below. Let us state our main results. We start from the case $p > 2$.

Theorem 2.1. *Assume that $p > 2$ and $\mu > 0$. Let $f \in L^2(Q)$, and let $u \in V_p(Q)$ be a weak solution of problem (1.1)–(2.2) or of problem (1.3)–(2.2). Then $u \in W^{2,2}(Q)$. Moreover, there is a constant c such that*

$$(2.8) \quad \|D^2 u\| \leq c \|f\|.$$

This theorem will be proved in the next section.

The other results concern the case $p < 2$. Note that, in this case, the parameter μ can be equal to zero, thus covering the p -Laplacian systems. Further, here we consider a general smooth bounded domain. On the other hand, we restrict our considerations to the full gradient case.

Before stating the regularity theorems for $p < 2$, let us recall two well known inequalities for the Laplace operator. The first, namely

$$(2.9) \quad \|D^2 v\| \leq C_4 \|\Delta v\|,$$

holds for any function $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Here $C_4 = C_4(\Omega)$. Note that if Ω is a convex domain, then $C_4 = 1$. For details we refer to [24] (Chapter I, estimate 20). The second kind of estimates which we are going to use for a $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $q \geq 2$, is

$$(2.10) \quad \|D^2 v\|_q \leq C_5 \|\Delta v\|_q,$$

where the constant C_5 depends only on q and Ω . It relies on standard estimates for solution of the Dirichlet problem for the Poisson equation. Actually, there are two constants K_1 and K_2 , independent of q , such that

$$(2.11) \quad K_1 q \leq C_5 \leq K_2 q.$$

Similarly, one has

$$(2.12) \quad \|v\|_{2,q} \leq C \|\Delta v\|_q,$$

where the constant C depends on q and Ω . For further details we refer to [23] and [34].

For $p < 2$ our main results are the following.

Theorem 2.2. *Let be $\mu \geq 0$, and $1 < p \leq 2$ such that $(2-p)C_4 < 1$, where C_4 is given by (2.9). Let $f \in L^{\frac{6}{p+1}}(\Omega)$. Then, the unique weak solution u of problem (1.1)-(2.1) belongs to $W^{2,2}(\Omega)$. Moreover, there is a constant C such that*

$$(2.13) \quad \|u\|_{2,2} \leq C \left(\|f\| + \|f\|^{\frac{1}{\frac{6}{p+1}}} \right).$$

If Ω is convex (or the cubic domain Q) the result holds for any $1 < p \leq 2$.

It is worth noting that in the limit case $p = 2$, when system (1.1) reduces to the Poisson equations, we recover the well known result

$$\|u\|_{2,2} \leq C \|f\|.$$

We set

$$(2.14) \quad C_6 = \max\{C_4, C_5\},$$

and

$$(2.15) \quad r(q) = \begin{cases} \frac{3q}{3 - (3-q)(2-p)} & \text{if } q < 3, \\ q & \text{if } q > 3. \end{cases}$$

Theorem 2.3. *Let be $\mu \geq 0$, $q > 2$, and $1 < p \leq 2$ such that $(2-p)C_6 < 1$, where C_6 is given by (2.14). Let $f \in L^{r(q)}(\Omega)$ and let u be the unique weak solution of problem (1.1)–(2.1). Then u belongs to $W^{2,q}(\Omega)$. Moreover, the following estimate holds*

$$(2.16) \quad \|u\|_{2,q} \leq C \left(\|f\|_q + \|f\|_{\frac{1}{r(q)}^{p-1}} \right).$$

Corollary 2.1. *Let p , μ and f be as in Theorem 2.3. Then, if $q > 3$, the weak solution of problem (1.1)–(2.1) belongs to $C^{1,\alpha}(\overline{\Omega})$, for $\alpha = 1 - \frac{3}{q}$.*

Note that, in (2.15), $r(q) > q$ for any $q < 3$. It is worth noting that $r(q)$ tends to the same value 3 as q tends to 3, from below and from above. Furthermore, if $q = 2$, the estimate (2.16) becomes simply (2.13). Finally, in estimates (2.13) and (2.16), the terms $\|f\|$ and $\|f\|_q$ can be replaced by 1.

Remark 2.1. *One could also consider the case where $f \in L^3(\Omega)$. We omit this further case and leave it to the interested reader. In this regard we stress that our interest mostly concerns the maximal integrability of the second derivatives of the solution.*

Remark 2.2. *When $p < 2$ we could extend to system (1.3) the regularity results up to the boundary obtained for system (1.1), by requiring a smallness condition on a suitable norm of f . Actually, following arguments already used in [12] and [13] for non-Newtonian fluids, the idea is to study the regularity for solutions of suitable approximating linear problems and then prove the regularity for solutions of the nonlinear problem, by employing the method of successive approximations. For brevity, here we avoid this further development.*

3 The $W^{2,2}(Q)$ -regularity: $p > 2$ and $\mu > 0$

In this section we prove Theorem 2.1. Therefore, throughout the section we work in the cubic domain Q . Let us introduce the definition of weak solutions of both the problems (1.1) and (1.3).

Definition 3.1. *Assume that $f \in V_p'(Q)$. We say that u is a weak solution of problem (1.1)–(2.2), if $u \in V_p(Q)$ satisfies*

$$(3.1) \quad \int_Q S(\nabla u) \cdot \nabla \varphi \, dx = \int_Q f \cdot \varphi \, dx,$$

for all $\varphi \in V_p(Q)$.

Definition 3.2. *Assume that $f \in V_p'(\Omega)$. We say that u is a weak solution of problem (1.3)–(2.2) if $u \in V_p(\Omega)$ satisfies*

$$(3.2) \quad \int_Q S(\mathcal{D}u) \cdot \mathcal{D} \varphi \, dx = \int_Q f \cdot \varphi \, dx,$$

for all $\varphi \in V_p(Q)$.

We recall that the existence and uniqueness of a weak solution can be obtained by appealing to the theory of monotone operators, following J.-L. Lions [26].

In proving Theorem 2.1 we focus on the symmetric gradient case, since the full gradient case is, in some respects, easier to handle. Hence we assume that S is given by

$$S(\mathcal{D}u) = (\mu + |\mathcal{D}u|)^{p-2} \mathcal{D}u,$$

with $\mu > 0$ and $p > 2$.

We follow arguments used in [6], in the context of non-Newtonian fluids. Therefore, we will try to preserve the notations. However in [6] (due to the divergence free constraint) the symbol $D_*^2 u$ has a slightly different meaning from that introduced in definition (2.3) below, since it also includes the derivatives $\partial_{33}^2 u_3$ (see (2.8) in [6]).

As in in [6], in order to avoid arguments already developed in other papers by the authors, we replace the use of difference quotients simply by differentiation.

It is an easy matter to obtain the following Korn's type inequality, proceeding, for instance, as in the proof given in [31].

Lemma 3.1. *There exists a constant C such that*

$$\|u\|_p + \|\nabla u\|_p \leq C \|\mathcal{D}u\|_p,$$

for all $u \in V_p(Q)$.

Lemma 3.2. *There exists a constant C such that*

$$\|D_*^2 u\|_p \leq C \|\nabla_* \mathcal{D}u\|_p,$$

for all $u \in V_p(Q)$.

This result reproduces Lemma 3.1 in [6], adapted to the new definition of $D_*^2 u$. Note that $\partial_s u = 0$ on Γ , $s = 1, 2$.

Actually, the above two lemmas hold for each $p > 1$.

Define, for $s = 1, 2$,

$$(3.3) \quad J_s(u) := \int_Q \nabla \cdot [(\mu + |\mathcal{D}u|)^{p-2} \mathcal{D}u] \cdot \partial_{ss}^2 u \, dx,$$

and

$$(3.4) \quad I_s(u) := \int_Q (\mu + |\mathcal{D}u|)^{p-2} |\partial_s \mathcal{D}u|^2 \, dx.$$

Lemma 3.3. *For any smooth function $u \in V_p(Q)$ the following inequality holds true*

$$(3.5) \quad J_s(u) \geq C_1 I_s(u),$$

with the constant C_1 given by (2.5).

Proof. Integrating twice by parts in (3.3) one gets

$$J_s(u) = \int_Q \partial_s \left[(\mu + |\mathcal{D}u|)^{p-2} \mathcal{D}u \right] \cdot \partial_s \nabla u \, dx.$$

Note that, due to symmetry, we replace $\partial_s \nabla u$ by $\partial_s \mathcal{D}u$. From the above expression, one has

$$J_s(u) = \int_Q \frac{\partial}{\partial D_{kl}} \left[(\mu + |D|)^{p-2} D_{ij} \right] \frac{\partial(\mathcal{D}u)_{kl}}{\partial x_s} \frac{\partial(\mathcal{D}u)_{ij}}{\partial x_s} \, dx,$$

where the derivatives with respect to D_{kl} are evaluated at the point $D = \mathcal{D}u$. Note that here we merely appeal to the chain rule. Then the result follows by using estimate (2.5). \square

Next we prove the following result which, roughly speaking, shows that the second tangential derivatives of u are square integrable.

Lemma 3.4. *Assume that $f \in L^2(Q)$ and let u be the solution of problem (1.3)–(2.2). Then $D_*^2 u \in L^2(Q)$ and*

$$(3.6) \quad \|D_*^2 u\| \leq \frac{c}{\mu^{p-2}} \|f\|.$$

Proof. Multiply both sides of the equations (1.1) by $\partial_{ss}^2 u$, $s = 1, 2$, and integrate over Q . By appealing to (3.3) and Lemma 3.3 it readily follows that

$$I_s(u) \leq c \|f\| \|\partial_{ss}^2 u\| \leq c \|f\| \|\nabla \partial_s u\|,$$

hence, from Lemma 3.1 applied to $\partial_s u$,

$$I_s(u) \leq c \|f\| \|\partial_s \mathcal{D}u\|.$$

Finally, observing that

$$\mu^{p-2} \|\partial_s \mathcal{D}u\|^2 \leq I_s(u),$$

one gets

$$\|\partial_s \mathcal{D}u\| \leq \frac{c}{\mu^{p-2}} \|f\|.$$

Application of Lemma 3.2, gives the result. \square

In order to complete the proof of Theorem 2.1 we have to show the integrability of the remaining second derivatives, namely the normal derivatives $\partial_{33}^2 u$. In doing this we follow the argument used in the paper [3]; we express these derivatives, pointwisely, in terms of the derivatives of u already estimated, and solve the corresponding system in the unknowns $\partial_{33}^2 u_i$, $i = 1, 2, 3$. Note that the main differences between this situation and that in reference [3], are the following: in [3] the L^2 -integrability of $\partial_{33}^2 u_3$ is known thanks to the divergence free constraint, $\partial_{33}^2 u_3 = -\partial_{31}^2 u_1 - \partial_{32}^2 u_2$. Hence the 3×3 linear system considered below is replaced, in [3], by a 2×2 linear system in the unknowns $\partial_{33}^2 u_i$, $i = 1, 2$. On the other hand, in reference [3], the presence of the pressure prevents the full $W^{2,2}$ -regularity.

For the missing derivatives we prove the following lemma.

Lemma 3.5. *Let u be the solution of problem (1.3)–(2.2). Then the vector field $\partial_{33}^2 u$ satisfies the pointwise estimate*

$$(3.7) \quad |\partial_{33}^2 u| \leq c \left(\frac{1}{\mu^{p-2}} |f| + |D_*^2 u| \right), \text{ a.e. in } Q.$$

Proof. Straightforward calculations show that

$$(3.8) \quad \begin{aligned} \partial_s [(\mu + |\mathcal{D} u|)^{p-2} \mathcal{D} u] &= (\mu + |\mathcal{D} u|)^{p-2} \partial_s \mathcal{D} u \\ &+ (p-2)(\mu + |\mathcal{D} u|)^{p-3} |\mathcal{D} u|^{-1} (\mathcal{D} u \cdot \partial_s \mathcal{D} u) \mathcal{D} u. \end{aligned}$$

For convenience, we set $\mathcal{D}_{jk} = (\mathcal{D} u)_{jk}$ and $B := (\mu + |\mathcal{D} u|)$. By using (3.8), the j^{th} equation (1.1), for any $j = 1, 2, 3$, takes the following form

$$(3.9) \quad B^{p-2} (\partial_{kk}^2 u_j + \partial_{jk}^2 u_k) + (p-2) B^{p-3} |\mathcal{D} u|^{-1} \mathcal{D}_{lm} \mathcal{D}_{jk} (\partial_{km}^2 u_l + \partial_{kl}^2 u_m) = -2f_j.$$

Let us write the previous three equations as a system in the unknowns $\partial_{33}^2 u_j$. For $j = 1, 2$ we have

$$(3.10) \quad B^{p-2} \partial_{33}^2 u_j + 2(p-2) B^{p-3} |\mathcal{D} u|^{-1} \mathcal{D}_{j3} \sum_{l=1}^3 \mathcal{D}_{l3} \partial_{33}^2 u_l = F_j - 2f_j,$$

where

$$(3.11) \quad \begin{aligned} F_j := & -B^{p-2} \sum_{k=1}^2 \partial_{kk}^2 u_j - B^{p-2} \sum_{k=1}^3 \partial_{jk}^2 u_k \\ & -2(p-2) B^{p-3} |\mathcal{D} u|^{-1} \sum_{\substack{l,m,k=1 \\ (m,k) \neq (3,3)}}^3 \partial_{km}^2 u_l \mathcal{D}_{jk} \mathcal{D}_{lm}. \end{aligned}$$

For $j = 3$ we have

$$(3.12) \quad 2B^{p-2} \partial_{33}^2 u_j + 2(p-2) B^{p-3} |\mathcal{D} u|^{-1} \mathcal{D}_{j3} \sum_{l=1}^3 \mathcal{D}_{l3} \partial_{33}^2 u_l = F_j - 2f_j,$$

where, for $j = 3$,

$$(3.13) \quad \begin{aligned} F_j := & -B^{p-2} \sum_{k=1}^2 \partial_{kk}^2 u_j - B^{p-2} \sum_{k=1}^2 \partial_{jk}^2 u_k \\ & -2(p-2) B^{p-3} |\mathcal{D} u|^{-1} \sum_{\substack{l,m,k=1 \\ (m,k) \neq (3,3)}}^3 \partial_{km}^2 u_l \mathcal{D}_{jk} \mathcal{D}_{lm}. \end{aligned}$$

The equations (3.10), for $j = 1, 2$, together with the equation (3.13) for $j = 3$ can be treated as a 3×3 linear system in the unknowns $\partial_{33}^2 u_j$, $j = 1, 2, 3$. Multiply all three equations by B^{2-p} . We denote the elements of the matrix

$A = A(x)$ associated with this system as a_{jl} , where $j, l = 1, 2, 3$. Then, we can write the system in a compact form as

$$(3.14) \quad a_{jl} \partial_{33}^2 u_l = G_j,$$

where the elements of the matrix of the system are given by

$$a_{jl} := \delta_{jl} + 2(p-2) (B |\mathcal{D} u|)^{-1} \mathcal{D}_{j3} \mathcal{D}_{l3},$$

for $j = 1, 2$, by

$$a_{jl} := 2\delta_{jl} + 2(p-2) (B |\mathcal{D} u|)^{-1} \mathcal{D}_{j3} \mathcal{D}_{l3},$$

and for $j = 3$, and

$$(3.15) \quad G_j := B^{2-p} (F_j - 2f_j).$$

Note that $a_{jl} = a_{lj}$; moreover, if ξ denotes any vector field then

$$a_{jl} \xi_j \xi_l = |\xi|^2 + \xi_3^2 + 2(p-2) (B |\mathcal{D} u|)^{-1} [\mathcal{D} u \cdot \xi]_3^2.$$

Hence, the matrix $A = (a_{jl})$ is also definite positive, a.e. in $x \in Q$, and the previous identity shows that

$$a_{jl} \xi_j \xi_l \geq |\xi|^2.$$

By setting $\xi = \partial_{33}^2 u$, we have obtained

$$(3.16) \quad |\partial_{33}^2 u|^2 \leq |G| |\partial_{33}^2 u|, \text{ a.e. in } Q,$$

where, obviously, by G we mean the vector (G_1, G_2, G_3) . Noting that, from (3.15), (3.11) and (3.13), there holds

$$(3.17) \quad |G_j| \leq \frac{2}{\mu^{p-2}} |f_j| + c |D_*^2 u|, \text{ a.e. in } Q,$$

from this estimate and (3.16) we get (3.5). \square

Finally, by combining (3.6) and (3.5) we readily obtain

$$\|D^2 u\| \leq \frac{c}{\mu^{p-2}} \|f\|$$

which is just (2.8). The proof of Theorem 2.1 is accomplished.

4 A regularity result for an approximating system: $p < 2$.

In the sequel we introduce an auxiliary positive parameter η and study the regularity for solutions of the following approximating problem

$$(4.1) \quad \begin{cases} -\eta \Delta v - \nabla \cdot S(\nabla v) = f, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

with S defined by (2.4), $\eta > 0$, $\mu > 0$ and $p \in (1, 2)$. The solutions v_η satisfy the estimate (4.15) below, with the constant C independent of η . This allows us to show that, as $\eta \rightarrow 0$, v_η tends, in a suitable sense, to the solution v of problem (4.1) with $\eta = 0$. A similar situation occurs, with respect to μ , as $\mu \rightarrow 0$.

We explicitly note that we introduce the above model just to approximate our solution by smooth functions.

Let us introduce the definition of weak solution of both the problems (4.1) and (1.1)–(2.1).

Definition 4.1. Assume that $f \in V_2'(\Omega)$. We say that v is a weak solution of problem (4.1) if $v \in V_2(\Omega)$ and satisfies

$$(4.2) \quad \eta \int_{\Omega} \nabla v \cdot \nabla \varphi \, dx + \int_{\Omega} S(\nabla v) \cdot \nabla \varphi \, dx = \int_{\Omega} f \cdot \varphi \, dx,$$

for all $\varphi \in V_2(\Omega)$.

Definition 4.2. Assume that $f \in V_p'(\Omega)$. We say that u is a weak solution of problem (1.1)–(2.1), if $u \in V_p(\Omega)$ satisfies

$$(4.3) \quad \int_{\Omega} S(\nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} f \cdot \varphi \, dx,$$

for all $\varphi \in V_p(\Omega)$.

As recalled in the previous section, the existence and uniqueness of a weak solution is known from the theory of monotone operators.

We start by proving the $W^{2,2}$ -regularity result stated in Proposition 4.1 below. In (4.4), the dependence of the constant c on Ω_0 , η and μ is omitted since the aim of the proposition is just to ensure that second derivatives are well defined a.e. in Ω . Following the notations introduced in section 2, by capital letters, C , C_1 , C_2 , etc., we denote positive constants independent of μ and of η also.

Proposition 4.1. Let $p \in (1, 2)$, $f \in L^2(\Omega)$, and v be a weak solution of problem (4.1). Then $v \in W_{loc}^{2,2}(\Omega)$ and, for any fixed open set $\Omega_0 \subset \subset \Omega$, there exists a constant c such that

$$(4.4) \quad \|D^2 v\|_{L^2(\Omega_0)} \leq c \|f\|.$$

Proof. As in the previous section, we formally use derivatives instead of difference quotients, to make the computation simpler. Fix an open set $\Omega_0 \subset \subset \Omega$. Let ζ be a $C_0^2(\Omega)$ -function, such that $0 \leq \zeta(x) \leq 1$ in Ω , and $\zeta(x) = 1$ in Ω_0 . Multiplying the first three equations in (4.1) by $-\nabla \cdot (\zeta^2 \nabla v)$ and integrating over Ω we get

$$(4.5) \quad \begin{aligned} & \eta \int_{\Omega} \partial_{jj}^2 v_i \partial_h (\zeta^2 \partial_h v_i) \, dx + \int_{\Omega} \partial_j [(\mu + |\nabla v|)^{p-2} (\nabla v)_{ij}] \partial_h (\zeta^2 \partial_h v_i) \, dx \\ & = - \int_{\Omega} f_i \partial_h (\zeta^2 \partial_h v_i) \, dx. \end{aligned}$$

By integration by parts, with respect to x_j and x_h , on the left-hand side one has

$$\begin{aligned}
(4.6) \quad & \eta \int_{\Omega} (\partial_{jh}^2 v_i)^2 \zeta^2 dx + \int_{\Omega} \partial_h [(\mu + |\nabla v|)^{p-2} (\nabla v)_{ij}] \partial_h (\nabla v)_{ij} \zeta^2 dx \\
& = -\eta \int_{\Omega} (\partial_{jh}^2 v_i) R_{ijh}(x) dx - \int_{\Omega} \partial_h [(\mu + |\nabla v|)^{p-2} (\nabla v)_{ij}] R_{ijh}(x) dx \\
& \quad - \int_{\Omega} f_i (\partial_{hh}^2 v_i) \zeta^2 dx - 2 \int_{\Omega} f_i (\partial_h v_i) \zeta (\partial_h \zeta) dx = \sum_{i=1}^4 I_i,
\end{aligned}$$

where, with obvious notation, R_{ijh} are lower order terms satisfying estimates

$$(4.7) \quad |R_{ijh}(x)| \leq c |\zeta| |\nabla \zeta| |\nabla v|.$$

As in the proof of Lemma 3.3, it is easy to verify, by appealing to (2.5), that

$$(4.8) \quad \int_{\Omega} \partial_h [(\mu + |\nabla v|)^{p-2} (\nabla v)_{ij}] \partial_h (\nabla v)_{ij} \zeta^2 dx \geq c \int_{\Omega} (\mu + |\nabla v|)^{p-2} |D^2 v|^2 \zeta^2 dx.$$

On the other hand, by Hölder's and Cauchy-Schwartz inequalities,

$$(4.9) \quad |I_1| \leq \epsilon \| |D^2 v| \zeta \|^2 + c(\epsilon) \|\nabla \zeta\|_{\infty}^2 \|\nabla v\|^2,$$

$$(4.10) \quad |I_3| \leq \epsilon \| |D^2 v| \zeta \|^2 + c(\epsilon) \|f\|^2,$$

and

$$(4.11) \quad |I_4| \leq c \|\nabla \zeta\|_{\infty} \|f\| \|\nabla v\|.$$

Further, by using the estimate

$$\frac{\partial S_{ij}(A)}{\partial A_{kl}} \leq c(\mu + |A|)^{p-2},$$

we have

$$(4.12) \quad |I_2| \leq c \int_{\Omega} (\mu + |\nabla v|)^{p-2} |D^2 v| |\zeta| |\nabla \zeta| |\nabla v| dx,$$

and, by the Cauchy-Schwartz inequality,

$$(4.13) \quad |I_2| \leq \epsilon \| |D^2 v| \zeta \|^2 + c(\epsilon) \|\nabla \zeta\|_{\infty}^2 \|\nabla v\|_2^2.$$

From (4.6) together with $\|\nabla v\| \leq c \|f\|$, it follows that

$$(4.14) \quad \| |D^2 v| \zeta \| \leq c \|f\|.$$

Hence (4.4) holds. \square

Our next step is to get a global estimate for the L^2 -norm of the second derivatives, uniform in η . This is the aim of the following proposition.

Proposition 4.2. *Let be $(2-p)C_4 < 1$, with C_4 given by (2.9). Let $f \in L^{\frac{6}{p+1}}(\Omega)$, and let v be a weak solution of problem (4.1). Then v belongs to $W^{2,2}(\Omega)$. Moreover, there exists a constant C such that*

$$(4.15) \quad \|v\|_{2,2} \leq C \left(\|f\| + \|f\|^{\frac{1}{\frac{6}{p+1}}} \right).$$

Proof. In order to avoid a useless dependence on μ , we assume, without loss of generality, $\mu \in (0, 1]$. At first note that, by replacing φ by v in (4.2) it is easy to get the following estimate for $\|\nabla v\|_p$, uniformly in η ,

$$\|\nabla v\|_p^p \leq \mu^p |\Omega| + 2^{2-p} \int_{\Omega} f \cdot v \, dx \leq C \left(1 + \int_{\Omega} f \cdot v \, dx \right).$$

Since, by Proposition 4.1, $v \in W_{loc}^{2,2}(\Omega)$, the i^{th} equation (4.1) can be written almost everywhere in Ω as

$$(4.16) \quad \begin{aligned} \eta \Delta v_i + (\mu + |\nabla v|)^{p-2} \Delta v_i \\ + (p-2)(\mu + |\nabla v|)^{p-3} |\nabla v|^{-1} \nabla v \cdot (\partial_j \nabla v) \partial_j v_i = -f_i. \end{aligned}$$

By multiplying both sides by Δv_i and summing over $i = 1, 2, 3$, we have

$$\begin{aligned} \eta |\Delta v|^2 + (\mu + |\nabla v|)^{p-2} |\Delta v|^2 \\ = (2-p)(\mu + |\nabla v|)^{p-3} |\nabla v|^{-1} \nabla v \cdot (\partial_j \nabla v) \partial_j v_i \Delta v_i - f_i \Delta v_i, \text{ a.e. in } \Omega. \end{aligned}$$

Next, we drop the term $\eta |\Delta v|^2$, and bound the left-hand side from below by $(\mu + |\nabla v|)^{p-2} |\Delta v|^2$. Multiplying the estimate thus obtained by $(\mu + |\nabla v|)^{2-p}$ and then integrating over Ω we get

$$\int_{\Omega} |\Delta v|^2 \, dx \leq (2-p) \int_{\Omega} |D^2 v| |\Delta v| \, dx + \int_{\Omega} (\mu + |\nabla v|)^{2-p} |f| |\Delta v| \, dx,$$

where we have used the estimate (for details see the Appendix)

$$|\nabla v \cdot (\partial_j \nabla v) (\partial_j v_i) \Delta v_i| \leq |\nabla v|^2 |D^2 v| |\Delta v|.$$

Observing that $(\mu + |\nabla v|)^{2-p} \leq \mu^{2-p} + |\nabla v|^{2-p}$, using Hölder's inequality, and dividing both sides by $\|\Delta v\|$, we get

$$(4.17) \quad \|\Delta v\| \leq (2-p) \|D^2 v\| + \|\nabla v\|^{2-p} \|f\| + \|f\|.$$

Let us estimate the first two terms on the right-hand side. For the first term we employ estimate (2.9). As far as the second term in (4.17) is concerned, by applying Hölder's inequality with exponents $3/(2-p)$ and $3/(p+1)$, the Sobolev embedding of $W^{2,2}(\Omega)$ in $W^{1,6}(\Omega)$, and by appealing to the estimate (2.12) with $q = 2$, we get

$$\|\nabla v\|^{2-p} \|f\| \leq \|\nabla v\|_6^{2-p} \|f\|_{\frac{6}{p+1}} \leq C \|\Delta v\|^{2-p} \|f\|_{\frac{6}{p+1}}.$$

By using the above estimates in (4.17), we get

$$\|\Delta v\| \leq (2-p) C_4 \|\Delta v\| + C \|\Delta v\|^{2-p} \|f\|_{\frac{6}{p+1}} + \|f\|.$$

Recalling that $(2-p)C_4 < 1$, and applying the Young's inequality

$$(4.18) \quad a^{2-p} b \leq \varepsilon a + c(\varepsilon) b^{\frac{1}{p-1}},$$

it is easy to recognize that the estimate

$$(4.19) \quad \|\Delta v\| \leq C \left(\|f\| + \|f\|^{\frac{1}{\frac{6}{p-1}}} \right)$$

holds. By using once again (2.12) we prove (4.15). \square

5 The $W^{2,2}$ -regularity result: $p < 2$.

Proof of Theorem 2.2. We deal separately with the case $\mu > 0$ and the degenerate case $\mu = 0$.

The case $\mu > 0$ - Consider the “sequence” (v_η) consisting of the solutions to problem (4.1), for $\eta > 0$. By the above proposition the sequence (v_η) is uniformly bounded in $W^{2,2}(\Omega)$. Therefore, by Rellich's theorem, there exists a field $u \in W^{2,2}(\Omega)$ and a subsequence, which we continue to denote by (v_η) , such that $v_\eta \rightharpoonup u$ weakly in $W^{2,2}(\Omega)$, and strongly in $W^{1,q}(\Omega)$ for any $q < 6$. Let us prove that

$$(5.1) \quad \int_{\Omega} S(\nabla u) \cdot \nabla \varphi \, dx = \lim_{\eta \rightarrow 0^+} \left\{ \int_{\Omega} S(\nabla v_\eta) \cdot \nabla \varphi \, dx + \eta \int_{\Omega} \nabla v_\eta \cdot \nabla \varphi \, dx \right\},$$

for any $\varphi \in C_0^\infty(\Omega)$. By applying (2.7) and then Hölder's inequality, we get

$$\begin{aligned} & \left| \int_{\Omega} S(\nabla u) \cdot \nabla \varphi \, dx - \int_{\Omega} S(\nabla v_\eta) \cdot \nabla \varphi \, dx \right| \\ & \leq c \int_{\Omega} (\mu + |\nabla u| + |\nabla v_\eta|)^{p-2} |\nabla u - \nabla v_\eta| |\nabla \varphi| \, dx \\ & \leq c \int_{\Omega} |\nabla u - \nabla v_\eta|^{p-1} |\nabla \varphi| \, dx \leq c \|\nabla v_\eta - \nabla u\|_p^{p-1} \|\nabla \varphi\|_p. \end{aligned}$$

The right-hand side of the last inequality tends to zero, as η goes to zero, thanks to the strong convergence of v_η to u in $W^{1,p}(\Omega)$. Further

$$\left| \eta \int_{\Omega} \nabla v_\eta \cdot \nabla \varphi \, dx \right| \leq \eta \|\nabla v_\eta\| \|\nabla \varphi\|,$$

where the right-hand side tends to zero as η goes to zero. Finally, observing that for any $\eta > 0$ and any $\varphi \in C_0^\infty(\Omega)$ the right-hand side of (5.1) is equal to $\int_{\Omega} f \cdot \varphi \, dx$, we show that u satisfies the integral identity (4.3) for any $\varphi \in C_0^\infty(\Omega)$. By a standard argument we show that u satisfies the integral equation (4.3), for any $\varphi \in V_p(\Omega)$. Hence u is a weak solution of (1.1), and belongs to $W^{2,2}(\Omega)$. Moreover, (2.13) follows from the relation $\|u\|_{2,2} \leq \liminf_{\eta \rightarrow 0^+} \|v_\eta\|_{2,2}$, together with (4.15). From the uniqueness of weak solutions we obtain the desired result.

The case $\mu = 0$ - Let us denote by u_μ the sequence of solutions of (1.1) for the different values of $\mu > 0$. We have shown that the sequence (u_μ) is

uniformly bounded in $W^{2,2}(\Omega)$. Therefore, exactly as above, we can prove the weak convergence of a suitable subsequence in $W^{2,2}(\Omega)$, and the strong convergence in $W^{1,q}(\Omega)$ for any $q < 6$, to the solution $u \in W^{2,2}(\Omega)$ of the problem (1.1) with $\mu = 0$. In this regard note that estimate (2.7) also holds with $\mu = 0$.

Finally we prove the last assertion in Theorem 2.2. For a smooth convex domain Ω estimate (2.9) holds with $C_4 = 1$. Hence the assumption on p is merely $p > 1$. \square

Remark 5.1. *We could adapt the above arguments to the case $p > 2$. Via a result similar to Proposition 4.1, one shows that the solution v of the approximated system 4.1 belongs to $W_{loc}^{2,2}(\Omega)$. Then reasoning as in the proof of Proposition 4.2, one obtains a global estimate for v in $W^{2,2}(\Omega)$, uniformly in η , with a restriction on the range of p , $p \in (2, 2 + \frac{1}{C_4})$, C_4 as in (2.9). Hence, as Theorem 2.2 above, one proves that the solution of (1.1), with $\mu > 0$, belongs to $W^{2,2}(\Omega)$. This result has the advantage to be directly proved in a general smooth domain, without need of localization techniques. However, it requires limitations on the range of p and, moreover, it cannot directly cover the case $\mu = 0$, since the $W^{2,2}(\Omega)$ -estimates that one obtain are not uniform in μ .*

6 The $W^{2,q}$ -regularity result: $q \geq 2$ and $p < 2$.

Proof of Theorem 2.3. From Theorem 2.2 we already know that the solution u of problem (1.1) belongs to $W^{2,2}(\Omega)$, since $(2-p)C_4 < 1$. Therefore, we can write equation (4.16) with u in place of v , and $\eta = 0$. By multiplying this equation by $(\mu + |\nabla u|)^{2-p}$, we can write, a.e. in Ω ,

$$(6.1) \quad -\Delta u - (p-2) \frac{\nabla u \cdot \nabla \nabla u \cdot \nabla u}{(\mu + |\nabla u|) |\nabla u|} = f (\mu + |\nabla u|)^{2-p},$$

where we have used the notation $\nabla u \cdot \nabla \nabla u \cdot \nabla u$ to denote the vector whose i^{th} component is $\nabla u \cdot (\partial_j \nabla u) \partial_j u_i = (\partial_i u_k) (\partial_j^2 u_k) (\partial_j u_i)$.

We start by proving an a priori L^q -estimate for the second derivatives of u by assuming, for the moment, that $u \in W^{2,q}(\Omega)$. We follow an argument similar to that used for proving the $W^{2,2}$ -estimates of u . We multiply both sides of equation (6.1) by $-\Delta u |\Delta u|^{q-2}$, and integrate in Ω . We get (for details see the Appendix)

$$\int_{\Omega} |\Delta u|^q dx \leq (2-p) \int_{\Omega} |D^2 u| |\Delta u|^{q-1} dx + \int_{\Omega} (\mu + |\nabla u|)^{2-p} |f| |\Delta u|^{q-1} dx.$$

By appealing to Hölder's inequality and to the inequality $(\mu + |\nabla u|)^{2-p} \leq 1 + |\nabla u|^{2-p}$, we show that

$$(6.2) \quad \begin{aligned} \|\Delta u\|_q^q &\leq (2-p) \|D^2 u\|_q \|\Delta u\|_q^{q-1} \\ &\quad + \|f\|_q \|\Delta u\|_q^{q-1} + \|\nabla u\|^{2-p} \|f\|_q \|\Delta u\|_q^{q-1}. \end{aligned}$$

Further, by dividing both sides by $\|\Delta u\|_q^{q-1}$, one gets

$$(6.3) \quad \|\Delta u\|_q \leq (2-p) \|D^2 u\|_q + \|f\|_q + \|\nabla u\|^{2-p} \|f\|_q.$$

We estimate the first term on the right-hand side of (6.3) via inequality (2.10).

Concerning the last term on the right-hand side, we start by assuming that $q \in (2, 3)$. As usual we denote by $q^* = 3p/(3-p)$ the Sobolev embedding exponent of q . By applying Hölder's inequality, with exponents $s = q^*/(2-p)q$ and $s' = r(q)/q$, we get

$$(6.4) \quad \| |\nabla u|^{2-p} f \|_q \leq \| \nabla u \|_{q^*}^{2-p} \| f \|_{r(q)}.$$

From (6.3), by appealing to (2.12), (6.4) and to Young's inequality one easily gets

$$\| \Delta u \|_q \leq (2-p) C_5 \| \Delta u \|_q + \| f \|_q + \varepsilon \| \Delta u \|_q + c(\varepsilon) \| f \|_{r(q)}^{\frac{1}{p-1}}.$$

Recalling the assumption on p , a further application of estimate (2.12) gives

$$(6.5) \quad \| u \|_{2,q} \leq C \left(\| f \|_q + \| f \|_{r(q)}^{\frac{1}{p-1}} \right).$$

Next we assume that $q > 3$. We will use arguments similar to the previous ones. Actually, by appealing to the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, to the estimate (2.12), and to Young's inequality, we estimate the last term on the right-hand side of (6.3) as follows:

$$(6.6) \quad \| |\nabla u|^{2-p} f \|_q \leq \| \nabla u \|_\infty^{2-p} \| f \|_q \leq C \| \Delta u \|_q^{2-p} \| f \|_q \leq \varepsilon \| \Delta u \|_q + c(\varepsilon) \| f \|_q^{\frac{1}{p-1}}.$$

Then, by repeating verbatim the arguments used above, one shows that u is bounded in $W^{2,q}(\Omega)$, uniformly with respect to μ , and that the estimate (6.5) holds. Finally, the argument used in the proof of the Theorem 2.2 in order to extend the results to the degenerate case $\mu = 0$ apply here as well.

The previous arguments are formal, since we have assumed that solutions belongs to $W^{2,q}(\Omega)$. However the following argument applies. Let us consider the problem

$$(6.7) \quad \begin{cases} -\Delta w^\varepsilon - (p-2) \frac{\nabla J_\varepsilon(u) \cdot \nabla \nabla w^\varepsilon \cdot \nabla J_\varepsilon(u)}{(\mu + J_\varepsilon(|\nabla u|)) J_\varepsilon(|\nabla u|)} = f (\mu + |\nabla u|)^{2-p}, & \text{in } \Omega, \\ w^\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

where w^ε is the unknown and J_ε denotes the Friedrichs mollifier. The coefficients of this modified system belong to $C^\infty(\mathbb{R}^n)$. We can also write this system in divergence form, as follows:

$$(6.8) \quad -\partial_h [m_{ijhk}(x) \partial_k w_j^\varepsilon] + (p-2) \partial_h [c_{ijhk}^\varepsilon(x) \partial_k w_j^\varepsilon] = f (\mu + |\nabla u|)^{2-p},$$

where

$$m_{ijhk}(x) = \delta_{ij} \delta_{hk} + (p-2) c_{ijhk}^\varepsilon(x)$$

and

$$c_{ijhk}^\varepsilon(x) = \partial_h J_\varepsilon(u_i) \partial_k J_\varepsilon(u_j) \frac{1}{(\mu + J_\varepsilon(|\nabla u|)) J_\varepsilon(|\nabla u|)}.$$

Further, let

$$c_{ijhk}(x) = (\partial_h u_i) (\partial_k u_j) \frac{1}{(\mu + |\nabla u|) |\nabla u|}.$$

From the well known estimate

$$(6.9) \quad |\nabla J_\varepsilon(u)| = |J_\varepsilon(\nabla u)| \leq J^\varepsilon(|\nabla u|)$$

we get

$$(6.10) \quad |c_{ijhk}^\varepsilon(x)| \leq 1, \quad \text{uniformly in } x, \varepsilon, \text{ and } \mu.$$

This shows that the system (6.8) (hence, the system (6.7)) is a linear elliptic system with regular coefficients. For such a system it is well known that if a force term F belongs to $L^q(\Omega)$, $q \geq 2$, then the solution belongs to $W^{2,q}(\Omega)$ (see, for instance, [19]). By following the previous arguments with u replaced by w^ε , and by using (6.9) and (6.10), it is straightforward to obtain the estimate (6.5) for w^ε . Note that such estimates are uniform with respect to μ and ε . Hence, there exists a subsequence, still denoted by w^ε , and an element $w \in W^{2,q}(\Omega)$ such that, as ε goes to zero, w^ε converges to w , weakly in $W^{2,q}$. Convergence is also strong in $W^{1,r}(\Omega)$: for any r if $q > 3$, and for any $r \in \left(1, \frac{3q}{3-q}\right)$ if $q < 3$. Let us show that w is a solution of the system

$$(6.11) \quad -\Delta w - (p-2) \frac{\nabla u \cdot \nabla \nabla w \cdot \nabla u}{(\mu + |\nabla u|) |\nabla u|} = f(\mu + |\nabla u|)^{2-p}.$$

To this purpose, we write equations (6.7) and (6.11) in the weak form, and take their difference, side by side. This leads to the expression

$$(6.12) \quad \begin{aligned} & \int_{\Omega} (\partial_h w_i^\varepsilon - \partial_h w_i) \partial_h \varphi_i \, dx + (2-p) \int_{\Omega} (c_{ijhk}^\varepsilon - c_{ijhk}) \partial_{hk}^2 w_j^\varepsilon \varphi_i \, dx \\ & + (2-p) \int_{\Omega} c_{ijhk} (\partial_{hk}^2 w_j^\varepsilon - \partial_{hk}^2 w_j) \varphi_i \, dx, \end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$. The first integral goes to zero as ε goes to zero, thanks to the strong convergence of w^ε to w in $W^{1,2}(\Omega)$. Concerning the second integral, we recall that mollifiers converge in L^p to the mollified function, as ε goes to zero, and that L^p convergence implies almost everywhere convergence of a subsequence. Therefore, c_{ijhk}^ε converges to c_{ijhk} , a.e. in Ω . From (6.10), by recalling that Ω is bounded and by using the dominated convergence theorem, it follows that

$$(6.13) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |c_{ijhk}^\varepsilon - c_{ijhk}|^2 \, dx = 0.$$

Hence the second integral in (6.12) goes to zero. The last integral in (6.12) tends to zero, thanks to the weak convergence of w^ε to w in $W^{2,q}(\Omega)$, since the coefficients $c_{ijhk}(x)$ are bounded.

Finally, it is easy to verify that $w = u$. Indeed, by taking the difference of (6.1) and (6.11), side by side, and by setting $V = u - w$, we get

$$\begin{cases} -\Delta V - (p-2) \frac{\nabla u \cdot \nabla \nabla V \cdot \nabla u}{(\mu + |\nabla u|) |\nabla u|} = 0, & \text{in } \Omega, \\ V = 0, & \text{on } \partial\Omega. \end{cases}$$

Finally, multiply the above equation by ΔV and integrate in Ω . By appealing to arguments already used, one readily recognizes that, under our assumptions on p , the vector V satisfies $\|\Delta V\| = 0$. Hence $V = 0$, by uniqueness. \square

The Corollary 2.1 is an immediate consequence of Theorem 2.3. Details are left to the reader.

7 Appendix

Our aim is to show the estimate

$$|I| := |\nabla v \cdot (\partial_j \nabla v) (\partial_j v_i) \Delta v_i| \leq |\nabla v|^2 |D^2 v| |\Delta v|.$$

In the sequel, for convenience, we sometimes avoid the summation convention, by explicitly writing the sums, even if repeated indexes appear.

We recall that

$$(D^2 v_k)^2 := \sum_{j,h=1}^3 |\partial_{jh}^2 v_k|^2 \quad \text{and} \quad |D^2 v|^2 := \sum_{k=1}^3 (D^2 v_k)^2 := \sum_{k,j,h=1}^3 |\partial_{jh}^2 v_k|^2.$$

We introduce the vectors b and w , whose components are defined as follows

$$b_j := (\partial_j v) \cdot \Delta v, \quad w_k^2 := \sum_{j,h=1}^3 ((\partial_h v_k) b_j)^2.$$

The modulus of vector b satisfies the following estimate:

$$|b| = \sum_{j=1}^3 b_j^2 \leq \sum_{j=1}^3 |\partial_j v|^2 |\Delta v|^2 = |\Delta v|^2 \sum_{j=1}^3 \sum_{i=1}^3 (\partial_j v_i)^2 = |\Delta v|^2 |\nabla v|^2.$$

Hence

$$(7.1) \quad w_k^2 = \sum_{h=1}^3 (\partial_h v_k)^2 \sum_{j=1}^3 b_j^2 = |\nabla v_k|^2 |\Delta v|^2 |\nabla v|^2.$$

Moreover

$$\begin{aligned} |I| &= \left| \sum_{j,h,k=1}^3 (\partial_h v_k) (\partial_{hj}^2 v_k) b_j \right| \leq \sum_{k=1}^3 \left| \sum_{j,h=1}^3 (\partial_{hj}^2 v_k) (\partial_h v_k) b_j \right| \\ &\leq \sum_{k=1}^3 \sqrt{\sum_{j,h=1}^3 (\partial_{hj}^2 v_k)^2} \sqrt{\sum_{j,h=1}^3 ((\partial_h v_k) b_j)^2}, \end{aligned}$$

where, in the last step, we have used that, for any pair of tensors A and B , there holds $|A \cdot B| \leq |A| |B|$. Hence, by the above notations and estimate (7.1), we get

$$\begin{aligned} |I| &\leq \sum_{k=1}^3 |D^2 v_k| |w_k| \leq |\Delta v| |\nabla v| \sum_{k=1}^3 |D^2 v_k| |\nabla v_k| \\ &\leq |\Delta v| |\nabla v| \sqrt{\sum_{k=1}^3 |D^2 v_k|^2} \sqrt{\sum_{k=1}^3 |\nabla v_k|^2} = |\Delta v| |\nabla v|^2 |D^2 v|, \end{aligned}$$

which is our thesis.

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